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On the k-normality of projected algebraic varieties

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Abstract. We give a necessary and sufficient condition for the isomorphic projection of a k-normal variety to remain k-normal, $k \geq 2$; the condition is based on a scheme $\mathrm{Z_k}$ naturally associated to degree k forms vanishing on the variety. We furnish many applications and examples especially in the case of varieties defined by quadratic equations. A non-vanishing theorem for the Koszul cohomology of projected varieties allows us to construct interesting examples in the last sections. All the results are effective and also interesting from the computational point of view.

Keywords: k-normality, projection, N_p conditions.

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Introduction

Let $\mathcal{L}$ be a very ample invertible sheaf on an irreducible algebraic variety X and let $V \subseteq \mathrm{H}^0(\mathcal{L})$ be a vector subspace defining an embedding $\phi_{|V|}\colon X \hookrightarrow \mathbb{P}(V) = \mathbb{P}^r$. We say that $X \subset \mathbb{P}^r$, or the linear system $|V|$, is *k-normal* if the restriction map

$$\mathrm{S^k}(V) = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to \mathrm{H}^0(\mathcal{O}_X(k)) = \mathrm{H}^0(\mathcal{L}^k)$$

is surjective. We say that the variety $X \subset \mathbb{P}^r$, or the linear system $|V|$, is *projectively normal* if X is a normal algebraic variety that is k-normal for every $k \geq 1$.

If $n = \dim(X)$ and $d = \deg(X) = \mathcal{L}^n$ (at least for $r \geq 2n+1$), one could expect that $X \subset \mathbb{P}^r$ is k-normal for $k \geq d + n - r$ (Castelnuovo bound). This

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statement was proved for (possibly singular) curves in [GLP] and for surfaces in [L], generalizing the classical results of Castelnuovo [C] for curves in $\mathbb{P}^3$, see also [K].

In this paper, we deal with the following problem: given an algebraic variety $X \subset \mathbb{P}^r$ that is k-normal, $k \geq 2$, and projecting it from a point $p \in \mathbb{P}^r$ such that the projection $\pi_p \colon X \to \mathbb{P}^{r-1}$ is an isomorphism (i.e., p is neither on any secant line to X nor on any tangent embedded space to X), when is the variety $Y = \pi_p(X) \subset \mathbb{P}^{r-1}$ still k-normal? Or, equivalently, do there exist conditions depending on X and V guaranteeing that a 1-codimensional very ample subsystem of a k-normal linear system $|V|$ is still k-normal?

If Y is k-normal, then, by Proposition 2.1,

$$h^0(\mathcal{I}_X(k)) = \binom{r+k-1}{r} + h^0(\mathcal{I}_Y(k)).$$

Hence $h^0(\mathcal{I}_X(k)) \geq \binom{r+k-1}{r}$, or equivalently $h^0(\mathcal{O}_X(k)) \leq \binom{r+k-1}{r-1}$.

For $k \geq 2$, we introduce a determinantal scheme $\mathrm{Z_k}(X)$ associated to the degree k forms vanishing on X. We prove that, if X is k-normal and if $h^0(\mathcal{I}_X(k)) \geq \binom{r+k-1}{r}$, then an isomorphic projection of X is k-normal if and only if the center of projection p does not belong to $\mathrm{Z_k}(X)$, to any secant line, or to any tangent embedded space (Theorem 2.6). In this way, we get a criterion that completely describes the 1-codimensional k-normal subsystems of a k-normal linear system. This criterion seems to have been unknown even for $k = 2$.

In Theorem 2.7, we show that, for a smooth algebraic variety with $h^0(\mathcal{I}_X(2)) \geq r+1$, we have $\mathrm{Sec}(X) \subseteq \mathrm{Z}_2(X)_{\mathrm{red}}$ (the variety of secant lines $\mathrm{Sec}(X)$ is a reduced scheme by definition). Since $\mathrm{Z}_2(X)$ is a determinantal scheme, this result is also interesting from a computational point of view in order to find a center of projection not lying on the secant variety. Moreover, this result gives an easy way of controlling 2-normality under projections. In particular, we prove that the projection of a smooth projectively normal variety is k-normal for every $k \geq 2$ if and only if the center of projection p does not belong to $\mathrm{Z}_2(X)$.

In Section 3, we focus on varieties cut out by quadratic equations. We study these quadratic equations as elements of linear systems on the ambient space and we look at the conditions assuring that $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$. For instance we apply Theorem 2.7 to show that the smooth projections into $\mathbb{P}^{r-1}$ of a smooth variety $X \subset \mathbb{P}^r$ satisfying property N_2 are k-normal for every $k \geq 2$ (Corollary 3.3).

Let us recall the following examples of varieties satisfying property N_2 to see the wide range of possible applications: varieties of minimal degree, smooth curves embedded by a line bundle of degree at least $2g + 3$, canonical curves whose Clifford index is bigger than 2, Segre varieties, Veronese embedding of $\mathbb{P}^n$, all embeddings of any projective variety given by a sufficiently ample line bundle and embeddings given by suitable adjoint line bundles. Applications and examples follow in abundance in Sections 3 and 4. Moreover, we remark that if $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$, then we have an effective method for writing down the equations of $\mathrm{Sec}(X)$.

In Section 4, we use Theorems 2.6, 2.7 and 3.2 to construct some counterexamples to Theorem 2.4, Corollary 2.6 and Theorem 3.1 of [B2], see Example 4.3, Example 4.4 and Example 4.5; these examples show that the "cohomological" vanishing theorems for Koszul cohomology work essentially only for projectively normal linear systems, where they were extensively used and applied (see [G], [GL]). In fact, we prove a non-vanishing theorem for the Koszul cohomology of a variety that is a smooth projection of a projectively normal variety defined by quadrics of small rank and of particular types (Theorem 4.2). Using this theorem, we also give a counterexample (Example 4.5) to Theorem 3.1 of [B1] by constructing, for every $d \geq 2$, an explicit cubic generator of the ideal of the projection into $\mathbb{P}^{N(d)-1}$, where $N(d) = \binom{d+2}{2} - 1$, of the Veronese embedding $X = \nu_d(\mathbb{P}^2) \subset \mathbb{P}^{N(d)}$.

Finally, we relate our results to known results on generic projections of $\nu_d(\mathbb{P}^2)$ into $\mathbb{P}^k$ where $k \geq 2d$, see [BE2]. Theorem 2.6 allows us to describe explicitly the k-normal linear subsystems. Furthermore, it yields an efficient way of constructing examples; in particular, by studying 2-normal projections of the Veronese surface into $\mathbb{P}^{2d}$ for every $d \geq 3$, we can construct a prime ideal of arbitrary codimension having a 3-linear resolution, i.e., generated by cubic forms and with syzygies generated by linear forms, answering to a question posed in [EG], p. 92.

1 Notations and definitions

Given an $r + 1$ dimensional vector space V be over $\mathbb{C}$, let $S = \oplus_{k=0}^{\infty} \mathrm{S}^{\mathrm{k}}(V)$ and $\mathbb{P}(V) = \mathbf{Proj}(S) = \mathbb{P}^r$. Let X be an algebraic variety, $\mathcal{L}$ a very ample invertible sheaf, and $V \subseteq \mathrm{H}^0(\mathcal{L})$ a linear subspace. Then V defines an embedding

$$\phi_{|V|} : X \hookrightarrow \mathbb{P}(V) = \mathbb{P}^r.$$

Note that $S = \oplus_{k \geq 0} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k))$ in this case. Abusing notation, we will use the same letter X also for $\phi_{|V|}(X)$. We say that $X \subset \mathbb{P}^r$, or the linear system $|V|$, is *k-normal* if the restriction map

$$\mathrm{S}^{\mathrm{k}}(V) = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to \mathrm{H}^0(\mathcal{O}_X(k)) = \mathrm{H}^0(\mathcal{L}^k)$$

is surjective.

We say that the variety $X \subset \mathbb{P}^r$, or the system $|V|$, is *projectively normal* if X is a normal algebraic variety that is k-normal for every $k \geq 1$. If V is a proper subspace of $\mathrm{H}^0(\mathcal{L})$, we call $\phi_{|V|}$ a *noncomplete embedding*, and $|V|$ a *noncomplete very ample linear system*. Note that a noncomplete embedding is the composition of the embedding defined by the complete linear system $|\mathcal{L}|$ with a linear projection.

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$, set $\Gamma_*(\mathcal{F}) := \oplus_{t \in \mathbb{Z}} \mathrm{H}^0(\mathcal{F}(t))$. Given a homogeneous ideal $I = \oplus_{t=0}^{\infty} I_t \subseteq S$, call $I^{sat} := \Gamma_*(\widetilde{I}) \subseteq \Gamma_*(\widetilde{S}) = S$ the *saturation* of I, where $\widetilde{I}$ and $\widetilde{S}$ are the associated sheaves. If $I = I^{sat}$, call I *saturated*. Given any positive integer m, call I *m-saturated* if $I_t = I_t^{sat}$ for every $t \geq m$.

Given any $X \subset \mathbb{P}^r$, set $I_X := \Gamma_*(\mathcal{I}_X)$, where $\mathcal{I}_X$ is the ideal sheaf of X. Set $S(X) := S/I_X$, which is the homogeneous coordinate ring of X; obviously $S(X)$ is a graded $\mathbb{C}$-algebra generated by its degree one part. Set $R(X) := \oplus_{k=0}^{\infty} \mathrm{H}^0(\mathcal{O}_X(k))$. There is a natural map $\gamma\colon S(X) \hookrightarrow R(X)$, which is an injective homomorphism of graded $\mathbb{C}$-algebras. Observe that, if $X \subset \mathbb{P}^r$ is k-normal for every $k \geq 1$, then the graded $\mathbb{C}$-algebra $R(X)$ is generated by its homogeneous degree-one part, since $S(X) \simeq R(X)$. That observation has this consequence, which we need.

Lemma 1.1. *Let $X \subset \mathbb{P}^r$ be an algebraic variety that is k-normal for every $k \geq 1$. Let $Y \subset \mathbb{P}^{r-1}$ be an isomorphic projection of X. If there exists a positive integer k_0 such that Y is k_0-normal, then Y is k-normal for every $k \geq k_0$.*

Proof. Let $\gamma\colon S(Y) \hookrightarrow R(Y)$ be the natural map. Since $R(Y) \simeq R(X) \simeq S(X)$, both $R(Y)$ and $S(Y)$ are finitely generated graded $\mathbb{C}$-algebras generated by their degree-one parts. We now use the following general fact: let $\gamma\colon S \hookrightarrow R$ be an injective homomorphism of graded $\mathbb{C}$-algebras generated by their degree-one parts; if there exists a k_0 for which $\gamma_{k_0}\colon S_{k_0} \hookrightarrow R_{k_0}$ is an isomorphism, then $\gamma_k\colon S_k \hookrightarrow R_k$ is an isomorphism for every $k \geq k_0$. $\square$

Let $m \in \mathbb{Z}$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is called *m-regular* if $\mathrm{H}^i(\mathcal{F}(m-i)) = 0$ for every $i > 0$, (see [Mu], p. 99). A homogeneous ideal $I \subseteq S$ is called *m-regular* if it is m-saturated and if the associated ideal sheaf $\widetilde{I}$ on $\mathbb{P}^r$ is m-regular. Let $I \subseteq S$ be a homogeneous ideal, and let

$$E_\bullet\colon 0 \to E_n \to E_{n-1} \to \ldots \to E_0 \to I \to 0$$

be any minimal graded free resolution of I. Recall that the following conditions are equivalent (see [E], p. 510):

1. I is m-regular;

2. E_i is the direct sum of modules $S(-j)$ with $j \leq m+i$ for $0 \leq i \leq n$.

In particular, a saturated m-regular homogeneous ideal $I \subseteq S$ is generated by forms of degrees m or less.

Let $I \subseteq S$ be a homogeneous ideal, M a finitely generated S-module. Say that I, respectively M, has a *p-linear resolution* over S if every minimal graded free resolution $E_\bullet$ of I, respectively of M, is such that $E_i = S(-p-i)^{\beta_i}$ for every $0 \leq i \leq n$ and for suitable integers β_i. In other words, I has a p-linear resolution if $I_t = 0$ for $t < p$, if I is generated by forms of degree p, and if all the maps of any minimal resolution are represented by matrices of linear forms. We will construct some examples with $p = 3$ at the end of the paper.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. If $\mathcal{F}$ is m-regular, then, by the Castelnuovo–Mumford lemma (see [Mu], p. 100),

1. $\mathcal{F}$ is $(m+1)$-regular, and

2. the map $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes \mathrm{H}^0(\mathcal{F}(m)) \to \mathrm{H}^0(\mathcal{F}(m+1))$ is surjective.

In the following proposition, we collect some simple consequences, which will be used below.

Proposition 1.2. *Let $\mathcal{L}$ be a very ample line bundle on a projective variety X such that $\mathrm{H}^{i+1}(\mathcal{L}^{1-i}) = 0$ for $0 \leq i \leq dim(X) - 1$, and let $|V| \subseteq |\mathcal{L}|$ be a very ample linear system. Set $r := dim(V) - 1$ and $X := \phi_{|V|}(X) \subset \mathbb{P}^r$. If X is 2-normal, then $\mathcal{I}_X$ is 3-regular; furthermore, then X is k-normal for every $k \geq 2$, and its ideal is generated in degree 3 or less.*

Proof. To prove $\mathcal{I}_X$ is 3-regular, we have to show $h^j(\mathbb{P}^r, \mathcal{I}_X(3-j)) = 0$ for $j \geq 1$. This vanishing results by considering the exact sequence

$$0 \to \mathcal{I}_X(3-j) \to \mathcal{O}_{\mathbb{P}^r}(3-j) \to \mathcal{O}_X(3-j) \to 0$$

with $j = i+2$. The last two assertions now follow from the Castelnuovo–Mumford lemma. □

We now want to recall the definition of the N_p property of a nondegenerate closed subvariety $X \subset \mathbb{P}^r$. Let $E_\bullet$ be a minimal free resolution of the S-module $R(X) = \oplus_{k \geq 0} \mathrm{H}^0(\mathcal{O}_X(k))$,

$$E_\bullet : 0 \to \oplus_j S(-j)^{\beta_{n,j}} \to \ldots \to \oplus_j S(-j)^{\beta_{0,j}} \to R(X) \to 0.$$

For any integer $p \geq 0$, we say X satisfies *property* N_p if $\beta_{0,0} = 1$, if $\beta_{0,j} = 0$ for $j > 0$ and if $\beta_{i,j} = 0$ for $1 \leq i \leq p$ and $j > i + 1$. Property N_0 means that X is k-normal for every $k \geq 1$. Property N_1 means that furthermore the homogeneous ideal of X is generated by quadratic forms. Property N_2 means that furthermore the syzygies among the quadratic generators are generated by linear forms, etc.

2 The main results

In this section $X \subset \mathbb{P}^r$ is a nondegenerate subvariety, π_p is the projection from a suitable point $p \in \mathbb{P}^r$ to a hyperplane avoiding p, and $Y := \pi_p(X) \subset \mathbb{P}^{r-1}$ is an isomorphic projection of X.

Assuming X is k-normal for some $k \geq 2$, we want to give a necessary and sufficient condition for Y to be k-normal. Let us begin with a necessary condition.

Proposition 2.1. *Let $X \subset \mathbb{P}^r$ be a k-normal variety, $k \geq 2$. If Y is k-normal in $\mathbb{P}^{r-1}$, then $h^0(\mathcal{I}_X(k)) = \binom{r+k-1}{r} + h^0(\mathcal{I}_Y(k))$. Hence $h^0(\mathcal{I}_X(k)) \geq \binom{r+k-1}{r}$, or equivalently $h^0(\mathcal{O}_X(k)) \leq \binom{r+k-1}{r-1}$, and equality holds if and only if $h^0(\mathcal{I}_Y(k)) = 0$.*

Proof. Since X is k-normal in $\mathbb{P}^r$, then

$$\begin{aligned} h^0(\mathcal{I}_X(k)) &= h^0(\mathcal{O}_{\mathbb{P}^r}(k)) - h^0(\mathcal{O}_X(k)) \\ &= h^0(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) + h^0(\mathcal{O}_{\mathbb{P}^{r-1}}(k-1)) + \cdots + h^0(\mathcal{O}_{\mathbb{P}^{r-1}}) - h^0(\mathcal{O}_Y(k)) \\ &= \binom{r+k-1}{r} + h^0(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) - h^0(\mathcal{O}_Y(k)) \\ &= \binom{r+k-1}{r} + h^0(\mathcal{I}_Y(k)), \end{aligned}$$

where the last equality follows from the k-normality of Y. □

We need the following definitions.

Definition 2.2. Let $X \subset \mathbb{P}^r$ be a variety with $h^0(\mathcal{I}_X(k)) =: \alpha \geq \binom{r+k-1}{r}$, and let $f_1, \ldots, f_\alpha$ be a basis of $\mathrm{H}^0(\mathcal{I}_X(k))$. Denote by $\mathrm{M_k}(X)$ the matrix of linear forms obtained by taking the $(k-1)$th partial derivatives of the f_i with respect to the coordinate functions. For any point $p \in \mathbb{P}^r$, denote by $\mathrm{M_k}(X)(p)$ the matrix obtained by evaluating $\mathrm{M_k}(X)$ at p. Finally, denote by $\mathrm{Z_k}(X)$ the (determinantal) scheme defined by the vanishing of the minors of maximal orders of $\mathrm{M_k}(X)$.

Remark 2.3. Obviously the definition of $\mathrm{Z_k}(X)$ does not depend on the choice of the base of $\mathrm{H}^0(\mathcal{I}_X(k))$. For $k = 2$, the scheme $\mathrm{Z}_2(X)$ was classically called the *Jacobian scheme* of the linear system $|\,\mathrm{H}^0(\mathcal{I}_X(2))|$. The Jacobian scheme of a linear system $|V|$ of hypersurfaces of degree k on $\mathbb{P}^r$ is defined by the vanishing of the minors of maximal order of the matrix of $(k-1)$ forms obtained by taking the first partial derivatives of the elements of a base of V.

If $h^0(\mathcal{I}_X(k)) \leq \binom{r+k-1}{r}$, the scheme $\mathrm{Z_k}(X)$ can also be defined in the same way. In this case $\mathrm{Z_k}(X)$ was studied classically, and its support is the locus of vertices of cones of degree k containing X. In our situation, the support of $\mathrm{Z_k}(X)$ is the locus of points p at which the space of these cones, passing through p, has dimension greater than expected; moreover, $\mathrm{Z_k}(X)$ is closely related to the k-normality of projections of X from these points, as we will see in Theorem 2.6.

Note that $\mathrm{M}_2(X)$ is the (homogeneous) Jacobian matrix of the rational map associated to the linear system of quadrics $|\,\mathrm{H}^0(\mathcal{I}_X(2))|$. This fact will allow us to deduce some interesting relations between $\mathrm{Sec}(X)$ and $\mathrm{Z}_2(X)$, see for example Theorem 2.7 and 3.2 below.

Remark 2.4. A point $p \in \mathbb{P}^r$ does not belong to $\mathrm{Z_k}(X)$ if and only if $\mathrm{rk}(\mathrm{M_k}(X)(p))$ is maximal, i.e., $\mathrm{rk}(\mathrm{M_k}(X)(p)) = \binom{r+k-1}{r}$.

We have the following result.

Proposition 2.5. *Let $X \subset \mathbb{P}^r$ be a variety with $h^0(\mathcal{I}_X(k)) =: \alpha \geq \binom{r+k-1}{r}$. Form the vector space of homogeneous polynomials of degree k defining cones containing $X \subset \mathbb{P}^r$ and having given vertex $p \in \mathbb{P}^r$. Then this vector space is of dimension $h^0(\mathcal{I}_X(k)) - \mathrm{rk}(\mathrm{M_k}(X)(p))$; moreover, if $Y := \pi_p(X) \subset \mathbb{P}^{r-1}$ is an isomorphic projection of X, then $h^0(\mathcal{I}_Y(k)) = h^0(\mathcal{I}_X(k)) - \mathrm{rk}(\mathrm{M_k}(X)(p))$.*

Proof. Let H^k be a hypersurface of degree k in $\mathbb{P}^r$. Then $\pi_p(H^k)$ is a hypersurface of degree k in $\mathbb{P}^{r-1}$ if and only if H^k is a cone with vertex passing through p, i.e., if and only if p is a point of multiplicity k on H^k. Hence $h^0(\mathcal{I}_Y(k))$ is the dimension of the vector space of polynomials of degree k defining cones with vertex through p and containing X.

We can calculate this dimension as follows: we have to determine the $a_1, \dots, a_\alpha$ such that $f = \sum_{i=1}^{\alpha} a_i f_i$ defines a hypersurface with a point of multiplicity k at p; i.e., the $(k-1)$th partial derivatives of f must vanish at p. Let

$$\frac{\partial^{k-1} f}{(\partial x_0)^{i_0} \dots (\partial x_r)^{i_r}} = 0,$$

be the corresponding system of $\binom{r+k-1}{r}$ equations in the unknowns $a_1, \dots, a_\alpha$, with $i_0 + \dots + i_r = k-1$. Set $a := (a_1, \dots, a_\alpha)$. We seek for the dimension of the space of solutions of the linear system of homogeneous equations given by

$$a \cdot \mathrm{M_k}(X)(p) = 0.$$

The result now follows via elementary linear algebra. □

Now we can prove a criterion of k-normality for the isomorphic projections of a k-normal variety.

Theorem 2.6. *Let $X \subset \mathbb{P}^r$ be a k-normal variety such that $h^0(\mathcal{I}_X(k)) \geq \binom{r+k-1}{r}$, or equivalently, $h^0(\mathcal{O}_X(k)) \leq \binom{r+k-1}{r-1}$, and let $p \in \mathbb{P}^r$ be such that the projection $\pi_p \colon X \to Y \subset \mathbb{P}^{r-1}$ is an isomorphism. Then $\pi_p(X) = Y \subset \mathbb{P}^{r-1}$ is a k-normal variety if and only if $p \notin \mathrm{Z_k}(X)$.*

In particular, if $|V|$ defines the embedding of a smooth k-normal variety X into $\mathbb{P}^r$, then every codimension-1 very ample subsystem of $|V|$ is k-normal if and only if $\mathrm{Z_k}(X)_{\mathrm{red}} \subseteq \mathrm{Sec}(X)$.

Proof. Proposition 2.5 says that the space of polynomials defining cones of degree k containing X, with vertex passing through p, has dimension $h^0(\mathcal{I}_Y(k))$, which is equal to $h^0(\mathcal{I}_X(k)) - \mathrm{rk}(\mathrm{M_k}(X)(p))$. Moreover, we obviously have

$$\mathrm{rk}(\mathrm{M_k}(X)(p)) \leq \binom{r+k-1}{r}.$$

Therefore, we have

$$\begin{aligned} h^0(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) - h^0(\mathcal{I}_Y(k)) &= \binom{r+k-1}{r-1} - h^0(\mathcal{I}_X(k)) + \mathrm{rk}(\mathrm{M_k}(p)) \\ &\leq \binom{r+k-1}{r-1} - h^0(\mathcal{I}_X(k)) + \binom{r+k-1}{r} \\ &= \binom{r+k}{r} - h^0(\mathcal{I}_X(k)) \\ &= h^0(O_{\mathbb{P}^r}(k)) - h^0(\mathcal{I}_X(k)) = h^0(\mathcal{O}_Y(k)), \end{aligned}$$

where the last equality holds because $\pi_p\colon X \to Y \subset \mathbb{P}^{r-1}$ is an isomorphism.

We conclude that Y is k-normal, i.e., that

$$h^0(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) - h^0(\mathcal{I}_Y(k)) = h^0(\mathcal{O}_Y(k)),$$

if and only if

$$\mathrm{rk}(\mathrm{M_k}(X)(p)) = \binom{r+k-1}{r},$$

i.e., if and only if $p \notin \mathrm{Z_k}(X)$ by Remark 2.4. □

Theorem 2.6 completely and explicitly describes the condition under which a very ample codimension-1 (or more) linear subsystem of a k-normal subsystem remains k-normal. In the literature, there are many results of this sort—for example, see [Me], [BE1], [BE2], [B1], [B2]. The difference in our result is that, with precision, we determine the open subset describing the k-normal very ample subsystems in the corresponding Grassmannian of subspaces of fixed codimension. Furthermore, our method is computational.

We now give a version of Theorem 2.6, which is useful in applications. This version was inspired by an unpublished algebraic result proved by Simis and Ulrich, relating the ideal of $\mathrm{Sec}(X)$ to some Fitting ideals of the equations defining X. Our proof is geometric and completely different.

Theorem 2.7. *Let $X \subset \mathbb{P}^r$ be a smooth algebraic variety. If $h^0(\mathcal{I}_X(2)) \geq r+1$, or equivalently $h^0(\mathcal{O}_X(2)) \leq \binom{r+1}{r-1}$, then* $\mathrm{Sec}(X) \subseteq \mathrm{Z_2}(X)_{\mathrm{red}}$.

In particular, if also X is 2-normal, then $Y := \pi_p(X) \subset \mathbb{P}^{r-1}$ is smooth and 2-normal if and only if $p \notin \mathrm{Z_2}(X)$. If also X is projectively normal, then Y is smooth and k-normal for every $k \geq 2$ if and only if $p \notin \mathrm{Z_2}(X)$.

Proof. Let us begin with the first part. Let $\phi\colon \mathbb{P}^r \dashrightarrow \mathbb{P}(\mathrm{H}^0(\mathcal{I}_X(2)))$ be the rational map defined by the linear system $|\,\mathrm{H}^0(\mathcal{I}_X(2))|$ of quadrics through X. Let $f_1, .., f_\alpha$ be a basis of $\mathrm{H}^0(\mathcal{I}_X(2))$ and set $W := V(f_1, \dots, f_\alpha)$. The rational map ϕ is defined exactly on $\mathbb{P}^r \setminus W$. We have the inclusion $W \subseteq \mathrm{Z_2}(X)$ because for every $p \in W$ the homogeneous system of linear equations

$$\mathrm{M_2}(X)(p) \cdot (x_0 \dots x_r)^t = 0$$

is satisfied by the coordinates of p by the Euler formula.

Take a point $p \in \mathrm{Sec}(X) \setminus W$, and let l_p be a (proper) secant line passing through p. The restriction of ϕ to l_p is given by a linear system of degree 2 having exactly two base points, possibly coincident, so that ϕ contracts l_p to a point. Therefore, $\mathrm{rk}(\mathrm{M}_2(X)(p))$ is not maximal by the sensitivity of the Jacobian of ϕ. Hence $p \in \mathrm{Z}_2(X)$ by Remark 2.4.

The second part follows from Theorem 2.6 and Lemma 1.1. □

It is computationally difficult to find a point outside the secant variety of a given variety $X \subset \mathbb{P}^r$. If $h^0(\mathcal{I}_X(2)) \geq r+1$, Theorem 2.7 says that it is sufficient to choose $p \notin \mathrm{Z}_2(X)$. This condition is easier to verify because $\mathrm{Z}_2(X)$ is an explicit determinantal scheme.

In [Me] the author produced some examples of projectively normal linear systems for which every codimension-1 subsystem is 2-normal (and hence k-normal for every $k \geq 2$ by Lemma 1.1). In these examples, the condition $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$ is always (and easily) verified. We will return to these examples in Corollary 3.3 and Theorem 4.2. In the following example, we show that, in general, the support of $\mathrm{Z}_2(X)$ is not contained in $\mathrm{Sec}(X)$.

Example 2.8. Let $X = \nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$ be the Veronese embedding of $\mathbb{P}^2$ by forms of degree 3. Then X is a projectively normal Del Pezzo surface of degree 9, whose ideal is generated by quadrics. Theorem 3.2 below assures that $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$. By Theorem 2.7, the smooth isomorphic projections $Y \subset \mathbb{P}^8$ are k-normal for every $k \geq 2$; i.e., every codimension-1 very ample subsystem of $|\mathcal{O}_{\mathbb{P}^2}(3)|$ is k-normal for every $k \geq 2$.

For such surfaces $Y \subset \mathbb{P}^8$, we can verify that $\mathrm{Z}_2(Y) \subsetneq \mathbb{P}^8$. Applying Theorem 2.7, we conclude that a general projection $W \subset \mathbb{P}^7$ of Y is k-normal for every $k \geq 2$. Here "general" means that the center of the projection does not belong to $\mathrm{Z}_2(Y)$.

Since W is 2-normal, we have $h^0(\mathcal{I}_W(2)) = 8 = \binom{7+1}{7}$. Looking at $\mathrm{M}_2(W)$, we see $\det(\mathrm{M}_2(W)) \not\equiv 0$. Hence $\mathrm{Z}_2(W)$ is a hypersurface of degree 8 in $\mathbb{P}^7$. Since $\mathrm{Sec}(W)$ is irreducible of dimension 5, we have $\mathrm{Sec}(W) \subsetneq \mathrm{Z}_2(W)_{\mathrm{red}}$. Projecting from a point $p \in \mathbb{P}^7 \setminus \mathrm{Z}_2(W)$, we obtain a smooth surface $S \subset \mathbb{P}^6$, which is k-normal for every $k \geq 2$ by Theorem 2.7. Proposition 1.2 yields that S is 3-regular and that the ideal of S is generated by cubic polynomials because $h^0(\mathcal{I}_S(2)) = 0$. The resolution of the ideal of S is then 3-linear because this ideal is 3-regular and it is generated by cubic polynomials.

If we project W from a point $p \in \mathrm{Z}_2(W) \setminus \mathrm{Sec}(W)$, then the projection will be smooth, but not 2-normal.

For smooth projections of $\nu_3(\mathbb{P}^2)$ into $\mathbb{P}^8$, the sharp Castelnuovo bound of [L] assures that these projections are k-normal for every $k \geq 3$, whereas we have shown they are also 2-normal. For smooth projections into $\mathbb{P}^7$ and $\mathbb{P}^6$, we can explicitly determine the 2-normal subsystems of a 2-normal $|V| \subset |\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(3))|$

of projective dimension 8.

3 Projected varieties that are k-normal for every $k \geq 2$

In this section, we study projectively normal varieties having smooth isomorphic projections that are k-normal for every $k \geq 2$. From now on, X will be smooth and nondegenerate in $\mathbb{P}^r$ of degree d and codimension s.

Let us recall the following definition introduced in [V], 2.2.

Definition 3.1. A subscheme $W \subset \mathbb{P}^r$ satisfies *condition* K_d if W is scheme theoretically cut out by forms $F_0, \dots, F_q$ of degree d such that the trivial (or Koszul) syzygies among the F_i are generated by the linear syzygies.

Theorem 3.2. *Let* $X \subset \mathbb{P}^r$ *be a smooth* 2*-normal variety such that* $\left(X, \mathrm{H}^0(\mathcal{I}_X(2))\right)$ *satisfies condition* K_2 *and such that* $\mathrm{Sec}(X) \subsetneq \mathbb{P}^r$*. Then every smooth projection of* X *into* $\mathbb{P}^{r-1}$ *is a* 2*-normal variety. In particular if* X *is also projectively normal, then every smooth isomorphic projection of* X *into* $\mathbb{P}^{r-1}$ *is a* k*-normal variety for every* $k \geq 2$*.*

Proof. By Theorem 2.7, it is sufficient to prove that $h^0(\mathcal{I}_X(2)) \geq r+1$ and that $\mathrm{Z}_2(X)_{\mathrm{red}} \subseteq \mathrm{Sec}(X)$. But these conditions hold since, if $\left(X, \mathrm{H}^0(\mathcal{I}_X(2))\right)$ satisfies condition K_2, then the associated rational map $\phi_{|\mathrm{H}^0(\mathcal{I}_X(2))|} : \mathbb{P}^r \dashrightarrow \mathbb{P}\left(\mathrm{H}^0(\mathcal{I}_X(2))\right)$ is an embedding off $\mathrm{Sec}(X)$; see [V], Corollary 2.5. □

Corollary 3.3. *Let* $X \subset \mathbb{P}^r$ *be a smooth variety satisfying condition* N_2 *and such that* $\mathrm{Sec}(X) \subsetneq \mathbb{P}^r$*. Then every smooth isomorphic projection of* X *is* k*-normal for every* $k \geq 2$*.*

Proof. It is immediate to see that condition N_2 implies condition K_2. □

The above results can be applied to a large class of examples: varieties of minimal degree, smooth curves embedded by a line bundle of degree greater than or equal to $2g+3$, canonical curves whose Clifford index is bigger than 2, Segre varieties, Veronese embeddings of $\mathbb{P}^n$, all sufficiently ample embeddings of any projective variety, suitable embeddings given by adjoint bundles. All these examples satisfy condition N_2, and every smooth isomorphic projection is then k-normal for every $k \geq 2$. (See [Me] for a different treatment made by explicit calculations in some of the above cases.)

Corollary 3.4. *Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety with* $\mathrm{Sec}(X) \subsetneq \mathbb{P}^r$. *Suppose $h^1(\mathcal{O}_X) = 0$ if $dim(X) \geq 2$. If $d \leq 2s - 1$, then X is arithmetically Cohen-Macaulay, and every smooth isomorphic projection of X into $\mathbb{P}^{r-1}$ is a k-normal variety for every $k \geq 2$.*

Proof. In proposition 2 of [AR], it is proved that, if X is as above, then it is arithmetically Cohen-Macaulay, and it satisfies condition N_2. We can now apply Corollary 3.3. □

The idea of verifying that the condition $\mathrm{Sec}(X) = \mathrm{Z}_2(X)_{\mathrm{red}}$ holds by looking at the rational map defined by $|\, \mathrm{H}^0(\mathcal{I}_X(2))|$ can be exploited also in the following interesting examples. Let us recall the definition of Severi varieties.

Definition 3.5. A smooth irreducible nondegenerate subvariety $X \subset \mathbb{P}^r$ is said to be a *Severi variety* if $\dim(X) = \frac{2}{3}(r-2)$ and $\mathrm{Sec}(X) \subsetneq \mathbb{P}^r$.

In [Z] Zak proved that there are only four Severi varieties:

1. the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$,
2. the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$,
3. the Plücker embedding of $\mathbb{G}(1, 5)$ in $\mathbb{P}^{14}$,
4. a 16 dimensional E_6 variety in $\mathbb{P}^{26}$.

The Severi varieties have the following uniform description. Let $\mathbb{A}_{\mathbb{R}}$ denote $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, i.e., the four real division algebras of real dimension, respectively, 1,2,4, 8. Let $\mathbb{A} = \mathbb{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and let $\mathcal{H}_{\mathbb{R}}$ denote the $\mathbb{A}_{\mathbb{R}}$-hermitian forms on $\mathbb{A}_{\mathbb{R}}^3$, i.e., the 3×3 $\mathbb{A}_{\mathbb{R}}$-hermitian matrices. If $x \in \mathcal{H}_{\mathbb{R}}$, then we may write

$$x = \begin{pmatrix} \alpha_1 & \overline{\beta}_1 & \overline{\beta}_2 \\ \beta_1 & \alpha_2 & \overline{\beta}_3 \\ \beta_2 & \beta_3 & \alpha_3 \end{pmatrix}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{A}_{\mathbb{R}}$. Let $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and let $X \subset \mathbb{P}(\mathcal{H})$ the locus of rank one elements.

The four Severi varieties are exactly $X \subset \mathbb{P}(\mathcal{H})$ for $\mathbb{A}_{\mathbb{R}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\mathrm{Sec}(X)$ is the locus of rank 2 matrices; clearly X is defined by $\dim(\mathbb{P}(\mathcal{H})) + 1$ quadrics. These quadrics define a rational map, $T : \mathbb{P}(\mathcal{H}) \dashrightarrow \mathbb{P}(\mathcal{H})$. One sees that T is a birational, involutory map that is an isomorphism on $\mathbb{P}(\mathcal{H}) \setminus \mathrm{Sec}(X)$. In fact, by writing down the equations defining X, one verifies that T is the composition of the map sending a matrix in $\mathcal{H}$ to the matrix of its cofactors and of an involutory projectivity of $\mathbb{P}(\mathcal{H})$. The cases of the Veronese surface and of the Segre 4-fold are classical (see for example [SR]).

A slightly different proof of the birationality of T is given in Theorem 2.5 of [ES], where the above Cremona transformations are characterized as the only *quadro-quadric* special Cremona transformations.

By the previous results we can prove the following proposition.

Proposition 3.6. *Let $X \subset \mathbb{P}^r$ be one of the four Severi varieties described above. Then* $\mathrm{Sec}(X) = (Z_2)_{\mathrm{red}}$, *and if $p \notin \mathrm{Sec}(X)$, then $Y = \pi_p(X) \subset \mathbb{P}^{r-1}$ is a smooth k-normal variety for every $k \geq 2$. Moreover, the ideals of the projected Veronese surfaces and of the projections of $\mathbb{P}^2 \times \mathbb{P}^2$ are generated by cubic forms and have* 3-*linear minimal resolutions.*

Proof. The Severi varieties are arithmetically Cohen Macaulay (see for example [Z], chapter III, theorem 1.2). As the quadrics defining them give rise to a rational map that is an isomorphism out of $\mathrm{Sec}(X)$, we get that $Z_2(X)_{\mathrm{red}} \subseteq \mathrm{Sec}(X)$, and the equality follows by Theorem 2.7. Then the last part of Theorem 2.7 implies that Y is k-normal for every $k \geq 2$. For the first two Severi varieties, the 2-normality of Y implies that $\mathcal{I}_Y$ is 3-regular and that the ideal of Y is generated by forms of degree less than or equal to three by applying Proposition 1.2. Since in both cases we have $h^0(\mathcal{I}_Y(2)) = 0$, the ideal of Y is generated by cubic forms and the minimal resolutions of $\mathcal{I}_Y$ are 3-linear. □

Remark 3.7. The fact that the projected Veronese surface in $\mathbb{P}^4$ is k-normal for every $k \geq 2$ can be also deduced from theorem 1 of [L]. We have preferred to give a uniform proof, which works for all the Severi varieties.

We remark that Theorem 3.1 of [B1] would imply that the ideal of the projection of $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^4$ is generated by quadrics. However, Proposition 3.6 shows that Theorem 3.1 of [B1] is incorrect.

The same idea can also be applied in the following examples, where the quadratic equations defining the varieties give rise to birational maps, but the varieties do not satisfy condition K_2.

Example 3.8. **(Projections of rational octic surfaces in $\mathbb{P}^6$).** Let $X \subset \mathbb{P}^6$ be an octic rational surface obtained as the embedding of the blowup of $\mathbb{P}^2$ at 8 points $p_1, \ldots, p_8$, no 4 on a line and no 7 on a conic, by the linear system of quartics containing the p_i. Then X is an arithmetically Cohen Macaulay surface, whose ideal is generated by 7 quadrics (see [HKS] and [MR]). The linear system $|\,\mathrm{H}^0(\mathcal{I}_X(2)|$ defines a special Cremona transformation of $\mathbb{P}^6$, and $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$ (see also [ST1] or [HKS]). Then every smooth projection of X into $\mathbb{P}^5$ is k-normal for every $k \geq 2$ by Theorem 2.7.

Example 3.9. (**Projections of septic elliptic scrolls of invariant** $e = -1$ **in** $\mathbb{P}^6$). Let $X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}^6$ be the septic elliptic scroll of invariant $e = -1$ whose hyperplane section is numerically equivalent to $C_0 + 3f$, where C_0 is the unique section with $C_0^2 = 1$ and f is the numerical class of a fiber. The scroll is a projectively normal surface whose ideal is generated by 7 quadrics defining a special Cremona transformation of $\mathbb{P}^6$ such that $\mathrm{Z}_2(X)_{\mathrm{red}} = \mathrm{Sec}(X)$ (see [ST2] or [HKS]). All smooth projections of X into $\mathbb{P}^5$ are k-normal for every $k \geq 2$ by Theorem 2.7.

4 A nonvanishing theorem for the Koszul cohomology of projections of varieties defined by quadrics of small rank

In [B2], the property N_p was generalized to the case of a variety embedded in $\mathbb{P}^r$ by an incomplete very ample linear system. Let $|V| \subseteq |\,\mathrm{H}^0(\mathcal{O}_X(1))|$ be a codimension-1 very ample linear subsystem. Let $Y := \phi_{|V|}(X) \subset \mathbb{P}^{r-1}$ be an isomorphic projection. Define the S-module $\widetilde{R(X)} = \oplus_{t\geq 0}\widetilde{R(X)}_t$ by letting $\widetilde{R(X)}_0 := \mathbb{C}$ and $\widetilde{R(X)}_1 := V$ and $\widetilde{R(X)}_t := \mathrm{H}^0(\mathcal{O}_X(t))$ for $t \geq 2$. Let $\widetilde{\beta}_{i,j}$ be the graded Betti numbers of a minimal resolution of $\widetilde{R(X)}$ as an S-module. For any integer $p \geq 0$, we say that V, or Y, *satisfies property* $\widetilde{N}_p$ if $\widetilde{\beta}_{0,0} = 1$, if $\widetilde{\beta}_{0,j} = 0$ for $j > 0$, and if $\widetilde{\beta}_{i,j} = 0$ for $1 \leq i \leq p$ and $j > i+1$.

Note that property $\widetilde{N}_0$ means that Y is k-normal for every $k \geq 2$; property $\widetilde{N}_1$ means that furthermore the homogeneous ideal of Y is generated by quadrics, etc. In the previous sections, we showed that, in many cases, a generic smooth projection of a variety X satisfies property $\widetilde{N}_0$ if X satisfies N_0. On the contrary, here we show that property N_p for $p \geq 1$ has bad behaviour under projection, even in the simplest cases.

First recall that property $\widetilde{N}_p$ can be characterized, as in [G], by the vanishing of certain Koszul cohomology groups. Recall the definition of the Koszul complex associated to V:

$$\to \Lambda^{p+1}V \otimes_{\mathbb{C}} S(-p-1) \to \Lambda^{p}V \otimes_{\mathbb{C}} S(-p) \to \Lambda^{p-1}V \otimes_{\mathbb{C}} S(-p+1) \to,$$

and of the homogeneous degree $(p+q)$-part of the twisted complex

$$\to \Lambda^{p+1}V \otimes_{\mathbb{C}} \widetilde{R(X)}_{q-1} \xrightarrow{d_{p+1,q-1}} \Lambda^{p}V \otimes_{\mathbb{C}} \widetilde{R(X)}_{q} \xrightarrow{d_{p,q}} \Lambda^{p-1}V \otimes_{\mathbb{C}} \widetilde{R(X)}_{q+1} \to$$

The *Koszul cohomology group* $\mathcal{K}_{p,q}(V, \widetilde{R(X)})$ is the cohomology group:

$$\mathcal{K}_{p,q}(V, \widetilde{R(X)}) := \frac{\ker(d_{p,q})}{\operatorname{im}(d_{p+1,q-1})}.$$

Recall the following two conditions are equivalent (see [B2], 2.2 and also [G]):

1. V satisfies the property $\widetilde{N}_p$
2. $\mathcal{K}_{i,q}(V, \widetilde{R(X)}) = 0$ for $0 \le i \le p$ and $q \ge 2$.

The previous equivalence tells us it is useful to study the Koszul cohomology groups associated to codimension-1 very ample subspaces $V \subset \mathrm{H}^0(\mathcal{O}_X(1))$. In particular, we will concentrate on the group $\mathcal{K}_{1,2}(V, \widetilde{R(X)})$, which is the cohomology in the middle of the following sequence of complex vector spaces:

$$\to \Lambda^2 V \otimes_{\mathbb{C}} V \xrightarrow{\alpha} V \otimes_{\mathbb{C}} \mathrm{H}^0(\mathcal{O}_X(2)) \xrightarrow{\beta} \mathrm{H}^0(\mathcal{O}_X(3)) \to 0.$$

Let us fix some more notation. Let $e_0, \dots, e_r$ be a base of the vector space $\mathrm{H}^0(\mathcal{O}_X(1))$ such that $e_0, \dots, e_{r-1}$ form a base of V. Then

$$S(V) = \mathbb{C}[x_0, \dots, x_{r-1}],$$

and we can identify V and the elements of degree 1 in $\mathbb{C}[x_0, \dots, x_{r-1}]$. We will use square brackets to indicate the classes modulo I_Y of the elements of $\mathrm{S}^{\mathrm{k}}(V)$. If Y is 2-normal and 3-normal, then every element of $\mathrm{H}^0(\mathcal{O}_X(2))$ can be written in the form $[\sum_{i,j} a_{i,j} e_i \otimes e_j]$ and every element in $\mathrm{H}^0(\mathcal{O}_X(3))$ has an expression of the form $[\sum_{i,j,k} b_{i,j,k} e_i \otimes e_j \otimes e_k]$.

Let us recall that by definition

$$\beta(e_i \otimes [e_j \otimes e_k]) = [e_i \otimes e_k \otimes e_j],$$

or equivalently, by the identification of S with $\mathbb{C}[x_0, \dots, x_{r-1}]$ we have

$$\beta(x_i \otimes [x_j x_k]) = [x_i x_j x_k].$$

By definition α operates in the following way:

$$\alpha((e_i \wedge e_j) \otimes e_k) = e_j \otimes [e_i \otimes e_k] - e_i \otimes [e_j \otimes e_k].$$

The conclusion is that, if Y is a 2-normal and 3-normal variety, then to show that $\mathrm{im}(\alpha) \subsetneq \ker(\beta)$, i.e., that $\mathcal{K}_{1,2} \neq 0$, it is sufficient to prove that in $\ker(\beta)$ there are some elements not of the following type

$$\sum a_{i,j,k}(x_j \otimes [x_i x_k] - x_i \otimes [x_j x_k]).$$

Below will use the following technical remark.

Remark 4.1. In the above notation, two elements $x_a \otimes [x_b x_c]$ and $x_d \otimes [x_e x_f]$ are equal in $V \otimes \mathrm{H}^0(\mathcal{O}_X(2))$ if and only if $x_a = x_d$ and $x_b x_c - x_e x_f \in I_Y$.

We proceed by proving the following theorem.

Theorem 4.2. *Let $X \subset \mathbb{P}^r$ be a k-normal variety, $k \geq 1$, whose ideal is generated by quadrics, i.e., X satisfies properties N_0 and N_1. Let $Y \subset \mathbb{P}^{r-1}$ be an isomorphic projection of X from the point $(x_r)_\infty = (0 : \ldots : 0 : 1)$ for which the property $\widetilde{N}_0$ holds. Suppose that, among the generators of the ideal I_X, there are two irreducible degree 2 polynomials of the form $x_h^2 - x_a x_r$ and $\phi_2 - x_h x_r$ with $\phi_2 \in \mathbb{C}[x_0, \ldots, x_{r-1}]_2$ and $h \neq r$, $a \neq r$, $a \neq h$ and that there is no irreducible polynomial vanishing on X of the form $x_h^2 - \psi_2$ with $\psi_2 \in \mathbb{C}[x_0, \ldots, x_{r-1}]_2$. Then $\mathcal{K}_{1,2}(V, \widetilde{R(X)}) \neq 0$, and hence there is at least one cubic generator of I_Y, namely, the cubic form $x_h^3 - x_a\phi_2$.*

Proof. As usual, we can identify V and $\mathbb{C}[x_0, \ldots, x_{r-1}]_1$. By eliminating x_r from the above degree-2 polynomials, we have $x_h^3 - x_a\phi_2 \in I_Y$; hence,

$$0 = [x_h^3 - x_a\phi_2] = \beta(x_h \otimes [x_h^2] - x_a \otimes [\phi_2]).$$

Thus $x_h \otimes [x_h^2] - x_a \otimes [\phi_2] \in \ker(\beta)$, but it is not in $\mathrm{im}(\alpha)$ as we will now show.

Let us suppose that

$$x_h \otimes [x_h^2] - x_a \otimes [\phi_2] = \sum_{i,j,k} a_{i,j,k}\{x_j \otimes [x_i x_k] - x_i \otimes [x_j x_k]\}$$

with $i, j, k \neq r$. By Remark 4.1, we have

$$x_h \otimes [x_h^2] = \sum a_{i,h,k} x_h \otimes [x_i x_k] - \sum a_{h,j,k} x_h \otimes [x_j x_k]$$

(recall that $a \neq h$). Therefore, we have

$$x_h^2 - \sum a_{i,h,k} x_i x_k + \sum a_{h,j,k} x_j x_k \in I_Y.$$

Since Y is the projection from $(x_r)_\infty$ of X, we would have

$$x_h^2 - \sum a_{i,h,k} x_i x_k + \sum a_{h,j,k} x_j x_k \in I_X,$$

in contradiction to the assumptions on the elements of I_X. □

Example 4.3. (**Projections of rational normal curves of degree** $r \geq 4$). Let $X \subset \mathbb{P}^r$ be a rational normal curve of degree $r \geq 4$. Denote the ideal of X by I_X. It is generated by the 2×2 minors of the $2 \times r$ matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{r-1} \\ x_1 & x_2 & \dots & x_r \end{pmatrix}.$$

The point $p = (0 : 1 : 0 : \dots : 0 : 1)$ does not belong to $\mathrm{Sec}(X)$. Modulo a projective transformation sending p into $(x_r)_\infty$, we can suppose that the equations defining X are given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 + x_r & \dots & x_{r-1} \\ x_1 + x_r & x_2 & \dots & x_r \end{pmatrix}$$

and that $(x_r)_\infty \notin \mathrm{Sec}(X)$. The hypothesis of Theorem 4.2 are satisfied: properties N_0 and N_1 hold for X; by Theorem 3.2 we know that Y satisfies property $\widetilde{N}_0$, where Y is the projection from $(x_r)_\infty$ of X; the assumptions on the generators of I_X are fullfilled by taking, for example, $h = r - 1$ and $a = r - 2$ and the two polynomials $x_{r-1}^2 - x_{r-2}x_r$ and $x_{r-1}(x_1 + x_r) - x_2 x_{r-2}$, which is $x_{r-1}x_1 - x_2 x_{r-2} + x_{r-1}x_r$; there are no polynomials of type $x_{r-1}^2 - \psi_2(x_0, \dots, x_{r-1})$ vanishing on X. Then, by Theorem 4.2, the ideal I_Y has a cubic generator of the following form:

$$x_{r-1}^3 - x_{r-2}(x_2 x_{r-2} - x_1 x_{r-1}).$$

We now construct some counterexamples to Theorem 3.1 of [B2] by using the above results.

Given a base point free linear system $|V| \subseteq |\mathcal{L}|$ on an algebraic variety X, we introduce the locally free sheaf M_V over X by the following exact sequence:

$$0 \to M_V \to V \otimes \mathcal{O}_X \to \mathcal{L} \to 0.$$

Let us recall the following assertions of [B2]:

1. Let $|V| \subseteq |\mathcal{L}|$ be a 2-normal very ample linear system on X, and $p \geq 0$ an integer. If $H^1(\Lambda^i M_V \otimes \mathcal{L}^k) = 0$ for $1 \leq i \leq p+1$ and $k \geq 1$, then $\phi_{|V|}(X) \subset \mathbb{P}(V)$ satisfies property $\widetilde{N}_p$ (Theorem 2.4 of [B2]).
2. Let X be a curve, and $|V| \subseteq |\mathcal{L}|$ a 2-normal very ample linear system on X. If $H^1(\Lambda^i M_V \otimes \mathcal{L}) = 0$ for some $p \geq 0$ and $1 \leq i \leq p+1$, then $\phi_{|V|}(X) \subset \mathbb{P}(V)$ satisfies property $\widetilde{N}_p$ (Corollary 2.6 of [B2]).
3. Let $|V| \subseteq |\mathcal{L}|$ be a very ample linear system on X satisfying property $\widetilde{N}_0$. If $H^1(\Lambda^2 M_V \otimes \mathcal{L}^j) = 0$ for $j \geq k-1$, then the ideal of $\phi_{|V|}(X) \subset \mathbb{P}(V)$ is generated by forms of degrees k or less (Theorem 3.1 of [B2]).

The following example is a counterexample to the previous assertions.

Example 4.4. Let $X \subset \mathbb{P}^r$ be a rational normal curve of degree $r \geq 4$ given by the equations at the end of Example 4.3. Let $V \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(r))$ be the very ample linear system associated to the projection from $(x_r)_\infty$ as above. Since V has no base points on X, we have an exact sequence of locally free sheaves

$$0 \to M_V \to V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(r) \to 0.$$

By construction $\mathrm{H}^0(V \otimes \mathcal{O}_{\mathbb{P}^1})$ injects into $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(r))$ giving $h^0(M_V) = 0$.

Since M_V is a rank $r-1$ vector bundle on $\mathbb{P}^1$, and since $\deg(M_V) = -r$ and $h^0(M_V) = 0$, we have the following splitting for M_V:

$$M_V \simeq \oplus_1^{r-2} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

Then $H^1(\Lambda^i M_V \otimes \mathcal{O}_{\mathbb{P}^1}(rm)) = 0$ for $i = 1, 2$ and for every $m \geq 1$. In the language of [B2], we say that $Y \subset \mathbb{P}^{r-1}$ possesses the property $\widetilde{N}_0$, but not the property $\widetilde{N}_1$, since its ideal requires at least a cubic generator, as was shown in Example 4.3.

In Proposition 3.6 we proved that the ideal of the projected Veronese surface $Y \subset \mathbb{P}^4$ is generated by cubic forms and that Y is k-normal for every $k \geq 2$, i.e., Y satisfies property $\widetilde{N}_0$ but not property $\widetilde{N}_1$, in opposition to Theorem 3.1 of [B1].

In the following example, we will show that there exist 1-codimensional very ample subspaces $V \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(d))$, for every $d \geq 3$, satisfying property $\widetilde{N}_0$, but not property $\widetilde{N}_1$, in opposition to Theorem 3.1 of [B1]. Moreover, we will produce an explicit cubic generator of the projected surface $Y_d = \phi_{|V|}(\mathbb{P}^2) \subset \mathbb{P}^{(d^2+3d-2)/2}$. These will be further counterexamples to Theorems 2.4 and 3.1 of [B2], they show that the vanishing of the cohomology groups associated to exterior powers of M_V does not imply the vanishing of $\mathcal{K}_{1,2}$.

Example 4.5. (**Projected Veronese surfaces of degree** $d \geq 3$). Let us consider the Veronese embedding of $\mathbb{P}^2$ by forms of degree d, $X_d = \nu_d(\mathbb{P}^2) \subset \mathbb{P}^{(d^2+3d)/2}$, $d \geq 3$. Then Corollary 3.4 implies that every 1-codimensional very ample subsystem $V \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(d))$ is 2-normal and that $Y_d = \phi_{|V|}(\mathbb{P}^2)$ is k-normal for every $k \geq 2$, i.e., Y_d satisfies property $\widetilde{N}_0$. In fact, $s = (d^2 + 3d - 4)/2$ and $d^2 \leq 2s - 1 = d^2 + 3d - 5$.

Let us take the distinct monomials of degree $d \geq 3$ in x, y, u as a basis of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and in $\mathbb{P}^{(d^2+3d)/2}$ let us take coordinates in the following way: $x_0 = u^d$, $x_1 = x^d$, $x_2 = y^d$, $x_3 = xu^{d-1}$, $x_4 = x^{d-1}u$, $x_5 = yu^{d-1}$, $x_6 = y^{d-1}u$, $x_7 = x^{d-1}y$, $x_8 = xy^{d-1}$ and so on. Among the elements of the ideal of X_d there are the quadratic polynomials:

$$x_0x_1 - x_3x_4,\ x_0x_2 - x_5x_6,\ x_1x_2 - x_7x_8.$$

One verifies that $p = (1 : 1 : 1 : 0 : \cdots : 0)$ does not belong to $\mathrm{Sec}(X)$, or equivalently, that $p \notin \mathrm{Z}_2(X_d)_{\mathrm{red}}$, which is computationally easier.

Apply the projective transformation

$$\omega : x'_0 = x_0,\ x'_1 = x_1 - x_0,\ x'_2 = x_2 - x_0,\ x'_j = x_j \text{ for } 3 \leq j \leq r,$$

and denote the new variables by $x_0, x_1, \ldots, x_r$. Then the polynomials

$$(x_0 + x_1)x_0 = x_3x_4,\ (x_0 + x_2)x_0 = x_5x_6,\ (x_0 + x_1)(x_0 + x_2) = x_7x_8$$

vanish on $\omega(X_d)$. Hence the quadratic polynomials

$$x_0x_2 + x_1x_2 - x_7x_8 + x_3x_4 \text{ and } x_0x_1 + x_1x_2 - x_7x_8 + x_5x_6$$

vanish on $\omega(X_d)$ too.

Project $\omega(X_d)$ from the point $p = (1 : 0 : 0 : 0 : \cdots : 0)$. We obtain a smooth projective surface Y'_d. The cubic polynomial

$$\psi := x_1x_7x_8 - x_1x_3x_4 - x_1^2x_2 - x_2x_7x_8 + x_2x_5x_8 + x_1x_2^2$$

vanishes on Y'_d because it is obtained by eliminating the variable x_0 between the two above quadratic polynomials vanishing on $\omega(X_d)$. We will show that ψ gives an element ψ' that is in $\ker(\beta)$, but not in $im(\alpha)$; i.e., it can be taken as a cubic generator of the ideal of Y'_d.

Let us remark that there are no quadratic elements in the ideal of $\omega(X_d)$ containing monomials in x_1^2, x_2^2 or x_1x_2 and not containing monomials in x_0x_j. Hence ψ gives rise to an element that is not in $im(\alpha)$.

Indeed, let us consider

$$\psi' := x_1\otimes[x_7x_8]-x_1\otimes[x_3x_4]-x_2\otimes[x_1x_1]-x_2\otimes[x_7x_8]+x_2\otimes[x_5x_6]+x_1\otimes[x_2x_2]$$

such that $\beta(\psi') = [\psi] = 0$. If ψ' were equal to an element of the form

$$\sum_{i,j,k} a_{i,j,k}\{x_j \otimes [x_i x_k] - x_i \otimes [x_j x_k]\}$$

with $i, j, k \neq 0$, then by remark 4 this expression would contain the term

$$-x_2 \otimes [x_1x_1] + x_1 \otimes [x_2x_1] + x_1 \otimes [x_2x_2] - x_2 \otimes [x_1x_2] + \cdots$$

and then we would have

$$\begin{aligned} x_1 \otimes [x_7x_8] - x_1 \otimes [x_3x_4] - x_2 \otimes [x_7x_8] + x_2 \otimes [x_5x_6] \\ = x_1 \otimes [x_2x_1] - x_2 \otimes [x_1x_2] + \cdots . \end{aligned}$$

But 4.1 tells us that also this relation is impossible, because there are no elements in the ideal of $\omega(X_d)$ that contain monomials in x_1x_2 and no monomials with x_0. By construction, any polynomial in the variables $x_1, \ldots, x_r$ vanishing on Y'_d gives a polynomial vanishing on $\omega(X_d)$ not containing the variable x_0.

Then, by looking back at the previous coordinates, we have that there exists a 1-codimensional very ample subsystem of $|\mathcal{O}_{\mathbb{P}^2}(d)|$ corresponding to hyperplanes through the point $p = (1 : 1 : 1 : 0 : \cdots : 0)$ that does not satisfy property $\widetilde{N}_1$.

In [BE2] generic projections of ruled and Veronese surfaces were studied. In the above terminology, we can rephrase Theorem 3 of [BE2] as follows: a general projection of $\nu_d(\mathbb{P}^2) \subset \mathbb{P}^{(d^2+3d)/2}$ into $\mathbb{P}^k$, where $k \geq 2d$, satisfies property $\widetilde{N}_0$.

As we pointed out before, Theorem 2.6 allows us to describe completely the open subset parameterizing very ample subsystems satisfying property $\widetilde{N}_0$ in the corresponding Grassmannians of subspaces of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(d))$. In fact, we can construct examples of surfaces $W_d \subset \mathbb{P}^{2d}$ of degree d^2 that are projections of $\nu_d(\mathbb{P}^2)$ and that are k-normal for every $k \geq 2$. It suffices, at every step, to take the center of projection outside the corresponding scheme Z_2; moreover, computationally the method is highly efficient. In particular, since $h^0(\mathcal{O}_{\mathbb{P}^2}(2d)) = h^0(\mathcal{O}_{\mathbb{P}^{2d}}(2))$ and since $\mathcal{I}_{W_d}$ is 3-regular by Proposition 1.2, I_{W_d} is generated by cubics, and the minimal resolutions of I_{W_d} are 3-linear according to Eisenbud-Goto (see [EG]). Hence we have an effective method for constructing prime ideals of arbitrary codimension that are 3-linear.

According to [EG], p. 92, it is particularly interesting to ask about domains of type S/I having p-linear resolution, with I homogeneous prime saturated ideal, i.e., to construct examples of algebraic varieties whose ideal is p-regular and generated by forms of degree p. Let us point out that, for any homogeneous ideal I, clearly $I_{\geq p} := \oplus_{t \geq p} I_t$ always has a p-linear resolution for p large enough, but it is neither saturated, nor prime.

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